W3L6 - POWER SERIES SOLUTION ABOUT SINGULAR POINTS

EX: Find the indicial root(s) and the corresponding recurrence relation for a series solution of $xy''+y'+2xy=0$ expanded about the regular singular point $x_0=0$.

Note: Method of Frobenius says we guess

$$y=\sum_{n=0}^{\infty} c_n (x-x_0)^{n+r} \qquad \text{Here } x_0=0$$

$$\Rightarrow y=\sum_{n=0}^{\infty} c_n x^{n+r}$$

Solution: $y=\sum_{n=0}^{\infty} c_n x^{n+r}$, $y'=\sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}$, $y''=\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}$

$$x\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}+\sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}+2x\sum_{n=0}^{\infty} c_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-1}+\sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}+\sum_{n=0}^{\infty} 2c_n x^{n+r+1}=0$$

Goal: Make exponent x^{k+r}, also need series to start at the same index

$$\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-1}+\sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}+\sum_{n=0}^{\infty} 2c_n x^{n+r+1}=0$$

$k=n-1$, $n=k+1$ $k=n-1$, $n=k+1$ $k=n+1$, $n=k-1$

$$\underbrace{\sum_{k=-1}^{\infty} c_{k+1}(k+1+r)(k+r)x^{k+r}}_{\text{evaluate at } k=-1,\ k=0}+\underbrace{\sum_{k=-1}^{\infty} c_{k+1}(k+1+r)x^{k+r}}_{\text{evaluate at } k=-1,\ k=0}+\sum_{k=1}^{\infty} 2c_{k-1}x^{k+r}=0$$

$$c_0(r)(r-1)x^{r-1}+c_1(r+1)rx^{r}+\sum_{k=1}^{\infty} c_{k+1}(k+1+r)(k+r)x^{k+r}$$

$$c_0(r)x^{r-1}+c_1(r+1)x^{r}+\sum_{k=1}^{\infty} c_{k+1}(k+1+r)x^{k+r}+\sum_{k=1}^{\infty} 2c_{k-1}x^{k+r}=0$$

$$c_0x^{r-1}\big(r(r-1)+r\big)+c_1x^{r}\big(r(r+1)+(r+1)\big)+\sum_{k=1}^{\infty}\left[c_{k+1}(k+1+r)(k+r)+c_{k+1}(k+1+r)+2c_{k-1}\right]x^{k+r}=0$$

Linear Independence will imply

assume $c_0\neq 0$ called the indicial root

$$c_0(r(r-1)+r)=0 \Rightarrow c_0r^2=0 \Rightarrow r^2=0 \Rightarrow r=0$$

but not always?

$$c_1(r+1)(r+1)=0 \Rightarrow c_1(r+1)^2=0 \Rightarrow c_1=0$$

$$c_{k+1}(k+1+r)(k+r)+c_{k+1}(k+1+r)+2c_{k-1}=0, \text{ for all } k\geq 1$$

Solve for c_{k+1} w/ $r=0$

$$c_{k+1}=\frac{-2c_{k-1}}{(k+1)^2},\quad k\geq 1 \leftarrow$$

$k=1 \qquad c_2 = \frac{-2c_0}{2^2} = -\frac{2c_0}{4} = -\frac{c_0}{2}$

$k=2 \qquad c_3 = \frac{-2c_1}{3^2} = 0$ all odd coeff $=0$ bc $c_1 = 0$

$k=3 \qquad c_4 = \frac{-2c_2}{4^2} = -\frac{2}{16}\left(\frac{-c_0}{2}\right) = \frac{c_0}{16}$

$k=5 \qquad c_6 = \frac{-2c_4}{6^2} = \frac{-2}{36}\left(\frac{c_0}{16}\right) = -\frac{c_0}{288}$

$y = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad r=0$

$= c_0 + \cancelto{0}{c_1} x + c_2 x^2 + \cancelto{0}{c_3} x^3 + c_4 x^4 + \cancelto{0}{c_5} x^5 + c_6 x^6 + \cdots$

$= c_0 - \frac{c_0}{2}x^2 + \frac{c_0}{16}x^4 - \frac{c_0}{288}x^6 + \cdots$